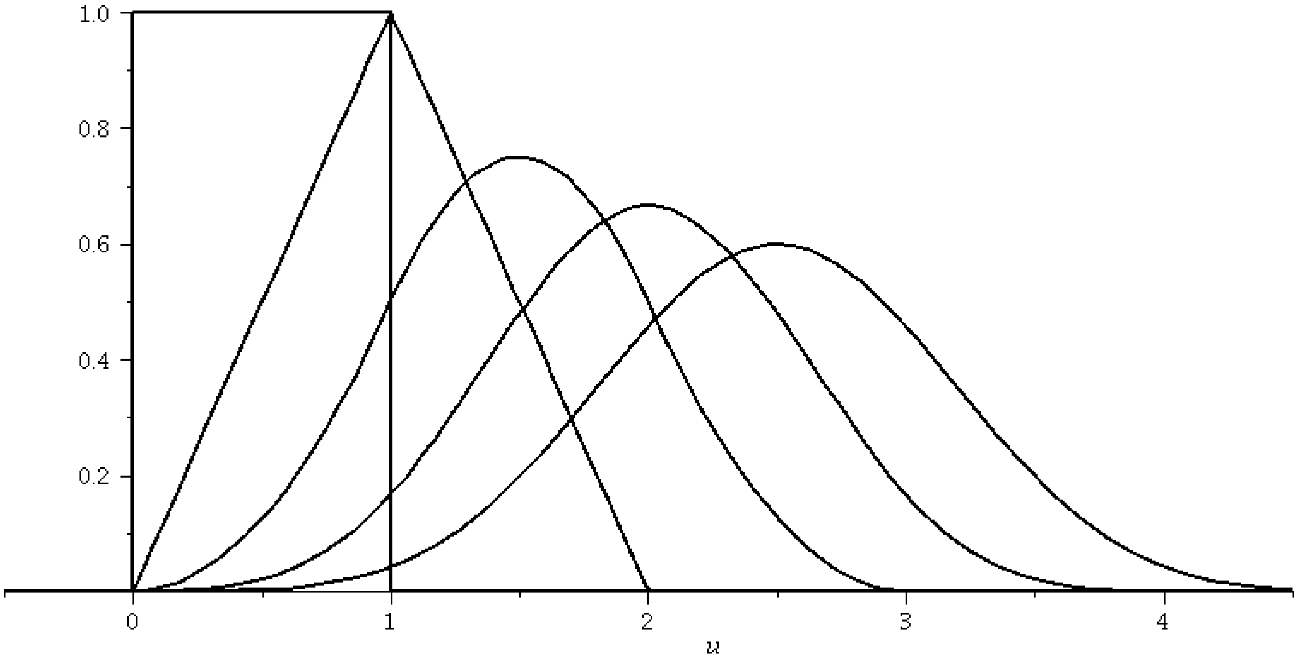
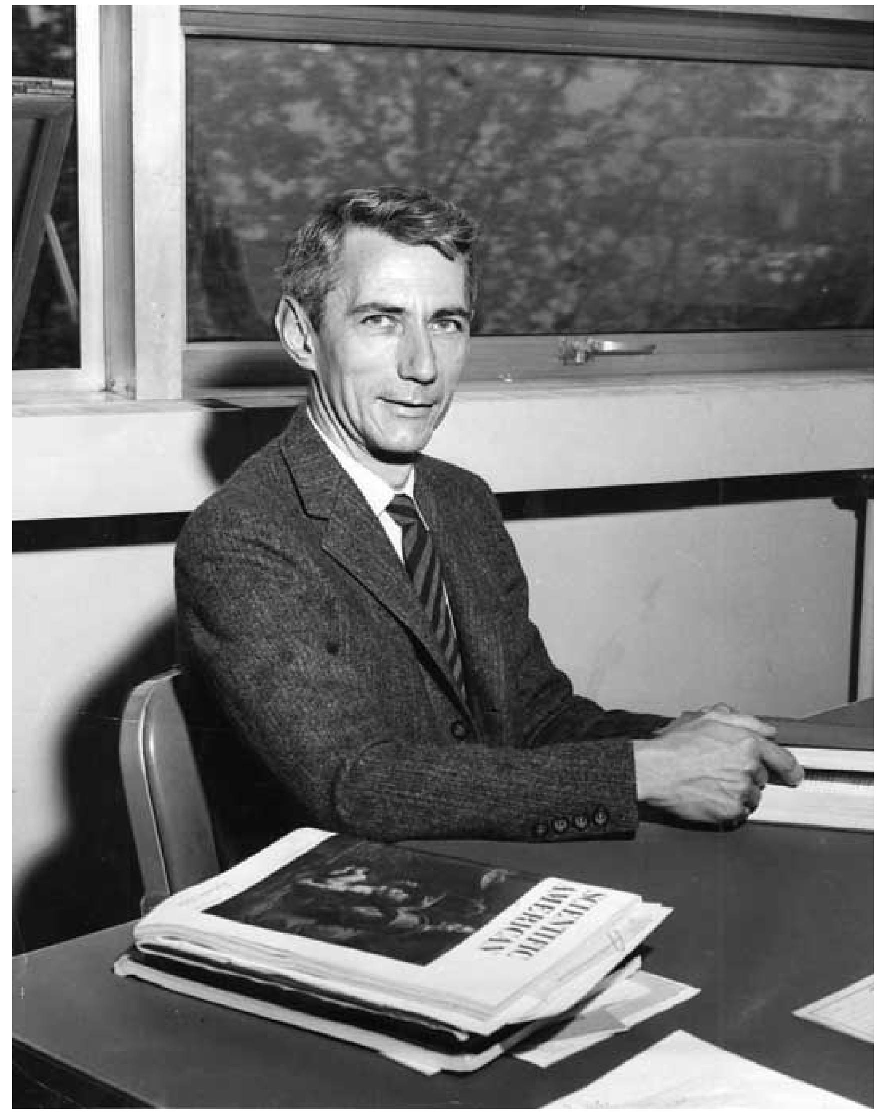
50 Years After Shannon



Claude E Shannon

- Working at Bell Labs, Shannon published his seminal paper “Communication in the Presence of Noise” in 1949.
- The paper heralded the industry of digital transmission of information, and Shannon went on to describe *entropy* and it's uses in cryptography at MIT.
- A lifelong chess fan, he also enjoyed juggling, unicycling and chess.



Nyquist-Shannon Sampling Theory

If a function $f(x)$ contains no frequency higher than ω_{max} (in radians/sec) it is completely determined by giving it's value at a series of points spaced $T = \pi / \omega_{max}$ seconds apart.

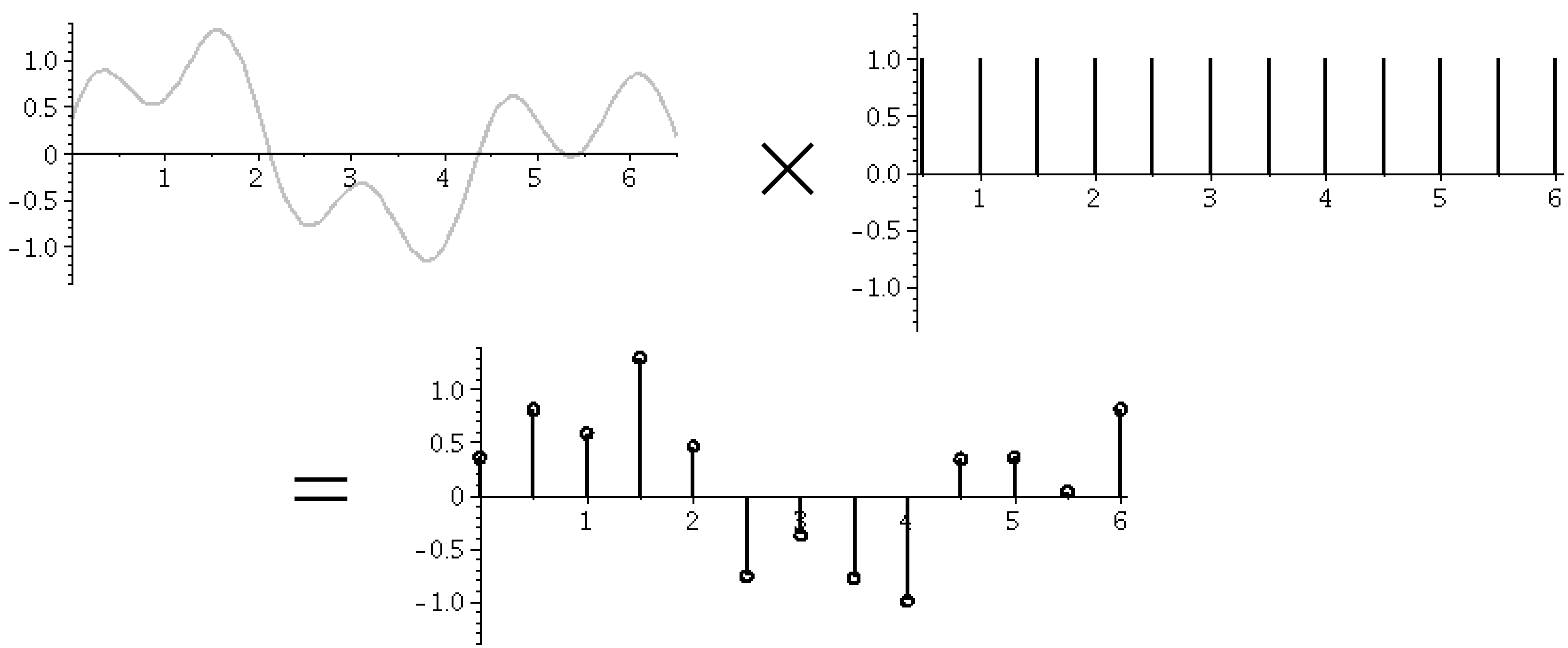
- The function must be *bandlimited*.
- It is *sampled* into discrete coefficients.
- It is *reconstructed* into a continuous signal.

Sampling

- We need the value of the function at regularly spaced, instants of time.

$$c[k] = \int_{-\infty}^{+\infty} \delta(x-k)\, f(x)\, dx$$

where $\delta(x)$ is the Dirac Delta function.



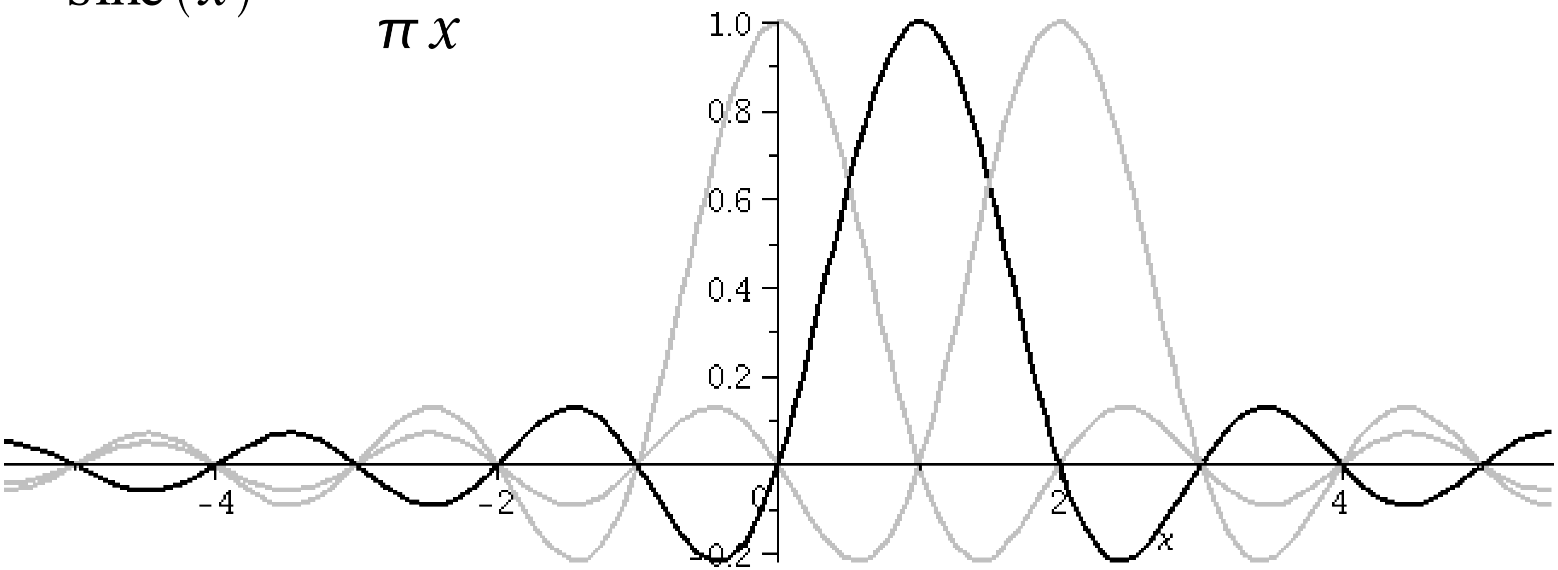
Reconstruction

- The points are reconstructed using a ideal sinc filter

$$f(x) = \sum_{k \in \mathbb{Z}} c[k] \operatorname{sinc}(x - k)$$

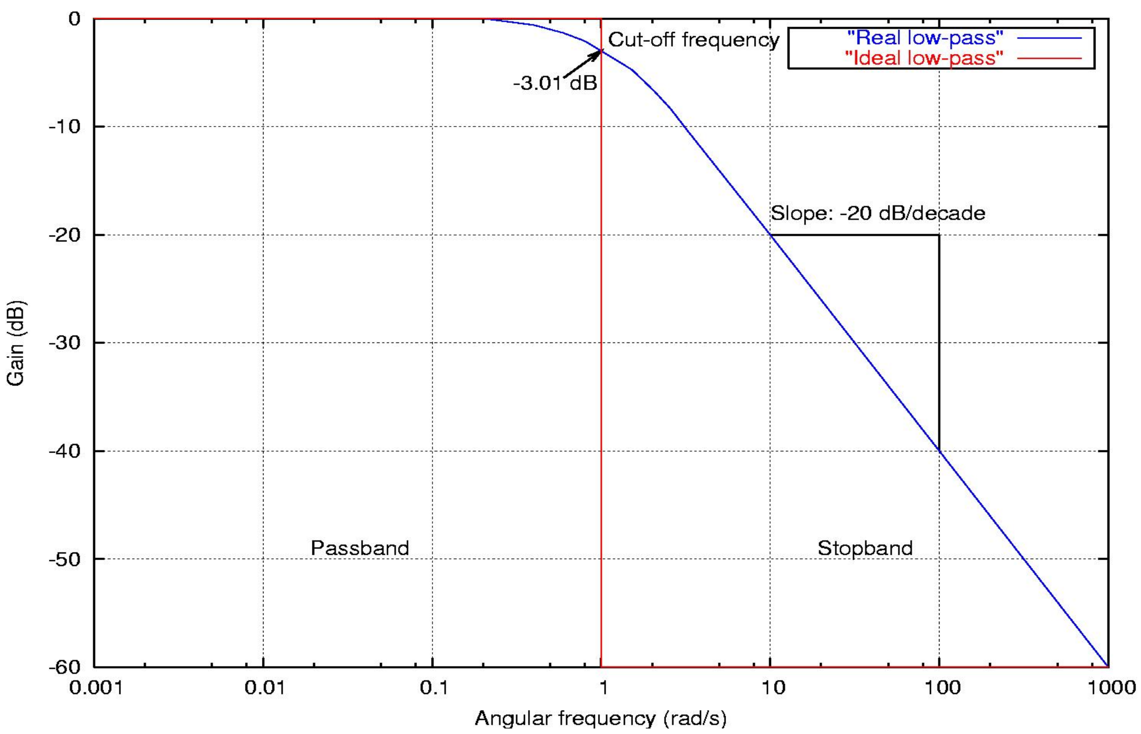
where:

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



Problems in the Real World

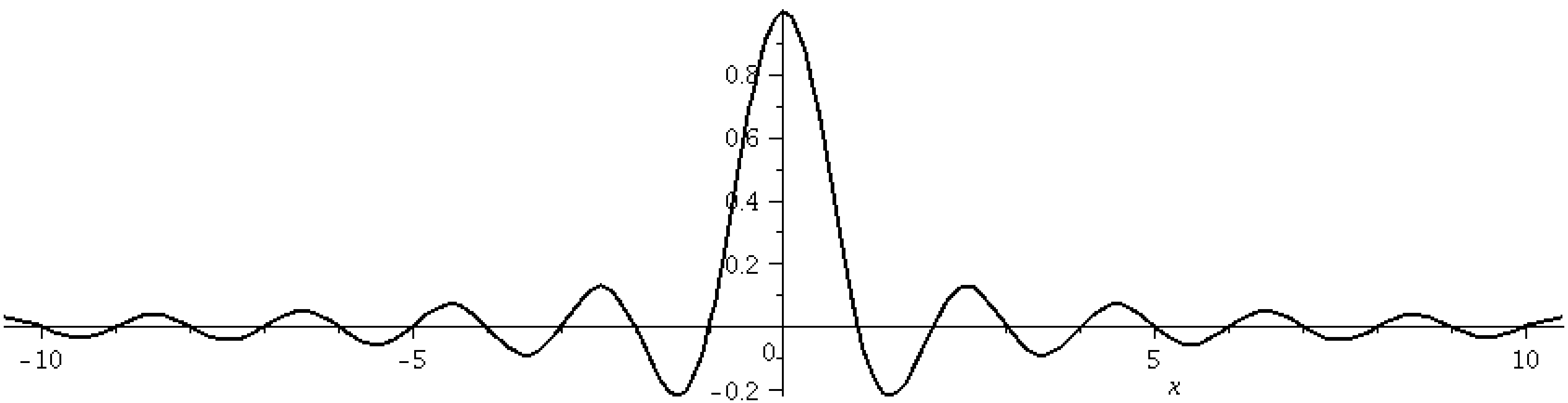
- There is no such thing as a truly bandlimited signal, so typically we filter the signal before sampling with a lowpass filter.



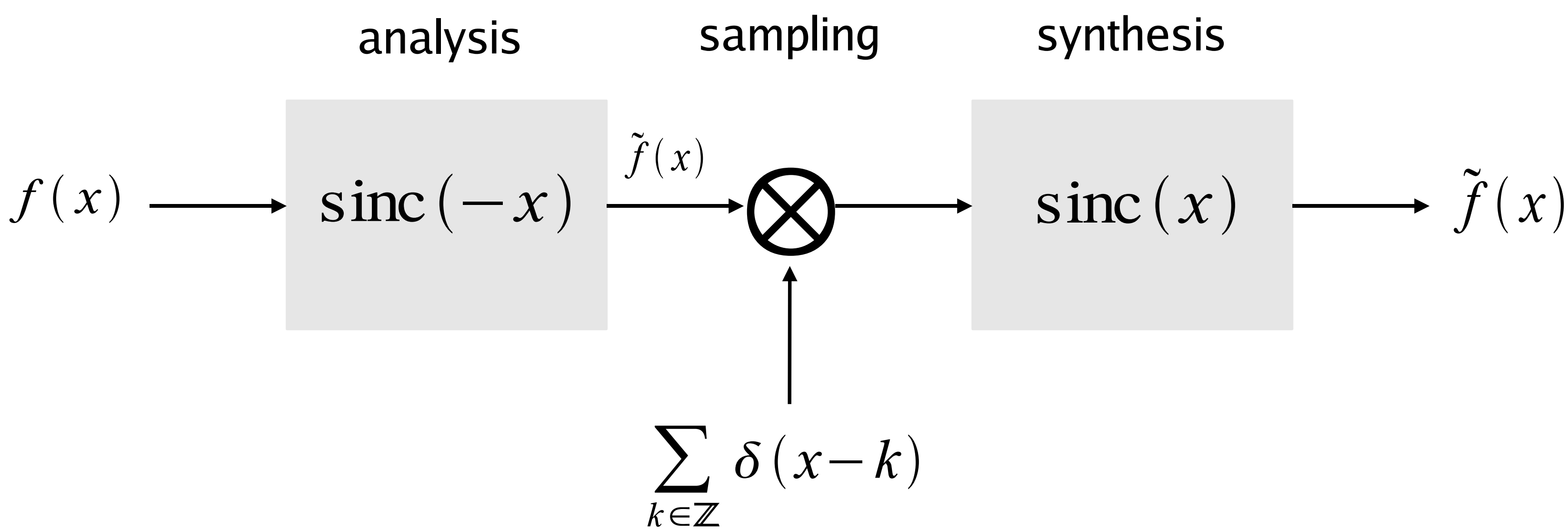
There are also no ideal lowpass filters.

Problems in the Real World

1. Shannon's reconstruction filter is rarely used in practice because of the slow decay of the sinc function.



Shannon's Sampling Reinterpreted



- Analysis $\tilde{f}(k) = \langle \mathrm{sinc}(x-k), f(x) \rangle$
- Synthesis $\tilde{f}(x) = \sum_{k\in\mathbb{Z}} \tilde{f}(k)\,\mathrm{sinc}(x-k)$
- sinc() is an Orthogonal basis: $\langle \mathrm{sinc}(x-k), \mathrm{sinc}(x-l) \rangle = \delta_{k-l}$

Formal Definitions

- Let's loosen the preconditions on our signal from *bandlimited* to just being a *finite energy function*, e.g. “square summable”.

  The Lebesgue's space of finite energy functions L_2:

  $$L_2(\mathbb{R}) = \left\{ f(x),\, x \in \mathbb{R} : \int_{x \in \mathbb{R}} |f(x)|^2 dx < +\infty \right\}$$

  The L_2 inner product:

  $$\langle f, g \rangle = \int_{x \in \mathbb{R}} f(x)\, g^*(x)\, dx$$

  The L_2 Norm:

  $$\|f\|_{L_2} = \left\{ \int_{x \in \mathbb{R}} |f(x)|^2 dx \right\}^{1/2} = \sqrt{\langle f, f \rangle}$$

Formal Definitions

- In the discrete world, this is known as a sequence in the l_2 space.

$$l_2(\mathbb{Z}) = \left\{ a[k], k \in \mathbb{Z} : \sum_{k \in \mathbb{Z}} |a[k]|^2 < +\infty \right\}$$

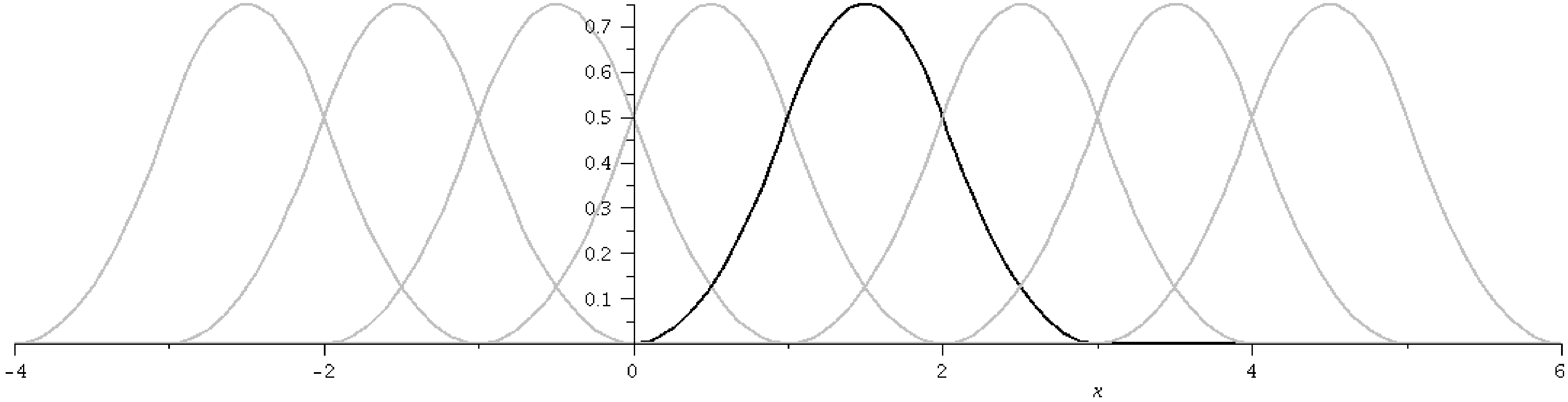
The l_2 norm:

$$\|a\|^2 = \left(\sum_{k \in \mathbb{Z}} |a[k]|^2 \right)^{1/2}$$

Shift Invariant Spaces

- An *integer shift invariant subspace* is the space of all functions that can be represented by summing weighted, integer offset copies of a basis function:

$$V(\varphi)=\left\{ f(x)=\sum_{k\in\mathbb{Z}} c[k]\varphi(x-k) \;\;:\;\; c\in l_2(\mathbb{Z}) \right\}$$



Fourier Transforms

- In continuous space the Fourier Transform is:

$$\hat{f}(\omega) = \int_{x \in \mathbb{R}} f(x)\, e^{-j\omega k}\, dx$$

- In discrete space:

$$A(e^{j\omega}) = \sum_{k \in \mathbb{Z}} a[k]\, e^{-j\omega k}$$

Gram Sequence

- The Gram or Autocorrelation Sequence measures how orthogonal a sequence of basis functions is with itself:

$$a_\varphi[k] = \langle \varphi(\cdot), \varphi(\cdot - k) \rangle$$

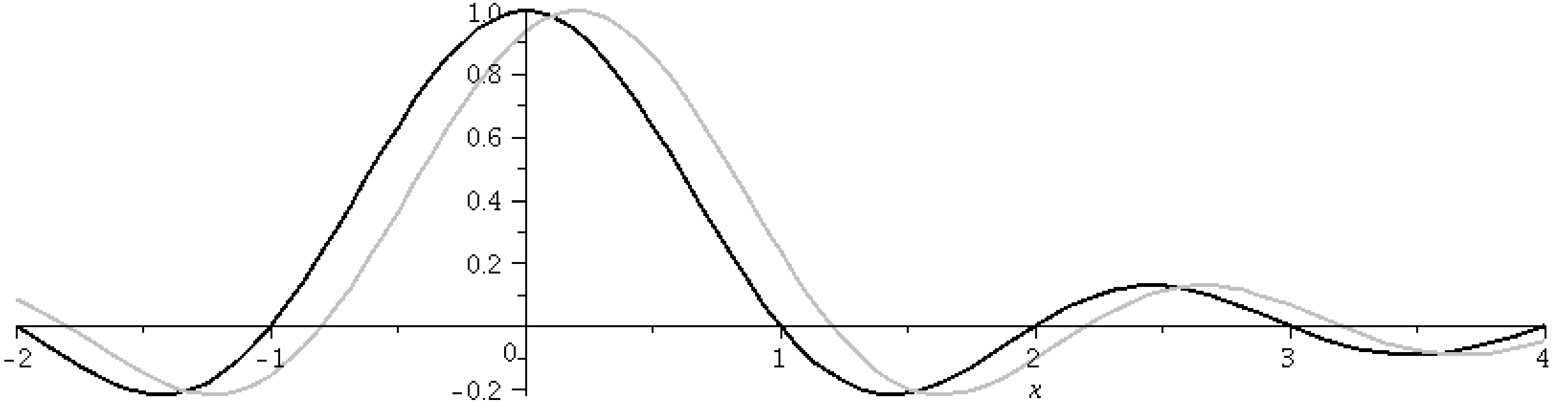
This also applies in frequency space:

$$A_\varphi(e^{j\omega}) = \sum_{n \in \mathbb{Z}} \left| \hat{\varphi}(\omega + 2\pi n) \right|^2$$

- Orthogonal basis functions display:

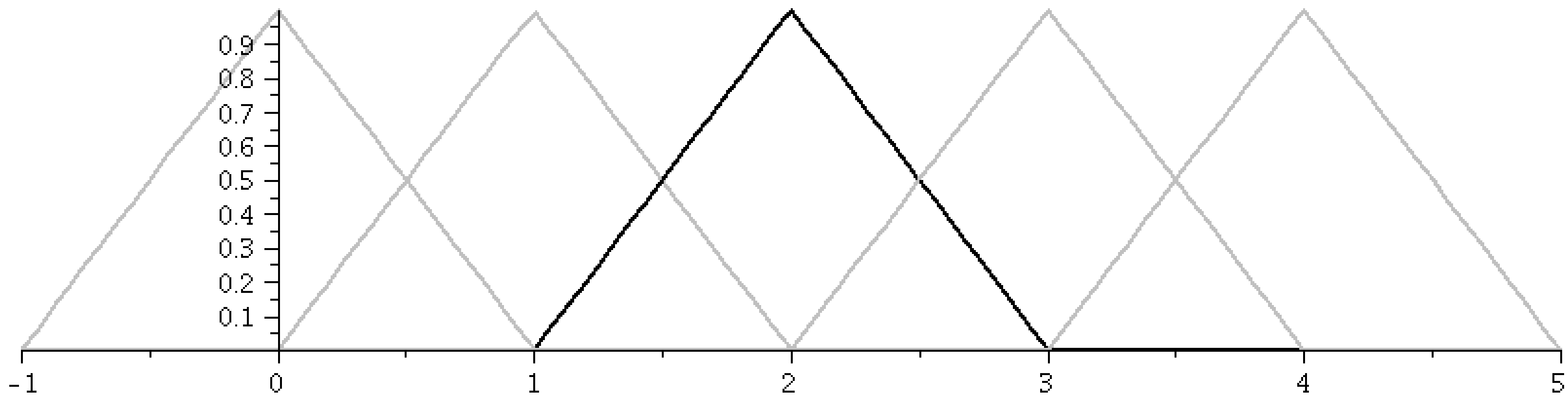
$$a_\varphi[k] = \delta_k \quad \Leftrightarrow \quad A_\varphi(e^{j\omega}) = 1 \quad \Leftrightarrow \quad \|c\|_{l_2} = \|f\|_{L_2}$$

Gram Sequence

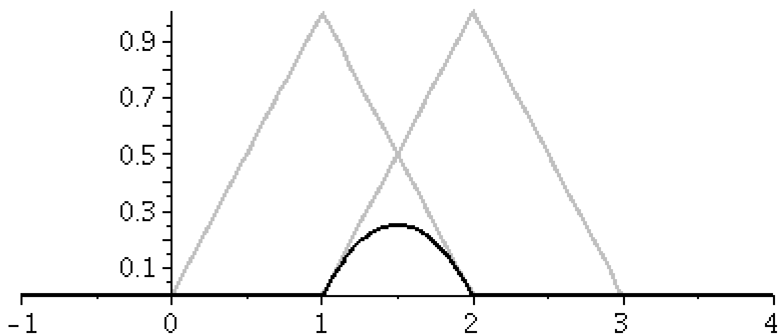


$$a_{\varphi}[k] = [\ldots 0, 0, 1, 0, 0, \ldots]$$

Gram Sequence



$$a_{\varphi}[k]=[\ldots 0,\frac{1}{6},1,\frac{1}{6,}0,\ldots]$$



Preconditions for Success

- Coefficients must be square summable.

  We proved that by specifying that $c[k]$ *must be in* l_2.

- The representation must be stable and unambigiously defined.

  We enforce this by asking that $\varphi_k = \varphi(x-k)$ *should form a Riesz Basis of* $V(\varphi)$

- The model should approximate a function as closely as desired by taking a smaller time step.

  This is enforced by the partition of unity condition:

$$\forall\, x \in \mathbb{R},\quad \sum_{k\in\mathbb{Z}} \varphi(x+k) = 1$$

Riesz Basis

- The Riesz Basis test says there are two values A and B that satisfy the condition:

$$\forall\, c[k] \in l_2,\quad A \cdot \|c\|_{l_2}^2 \;\le\; \left\| \sum_{k\in\mathbb{Z}} c[k]\varphi_k \right\|^2 \;\le\; B \cdot \|c\|_{l_2}^2$$

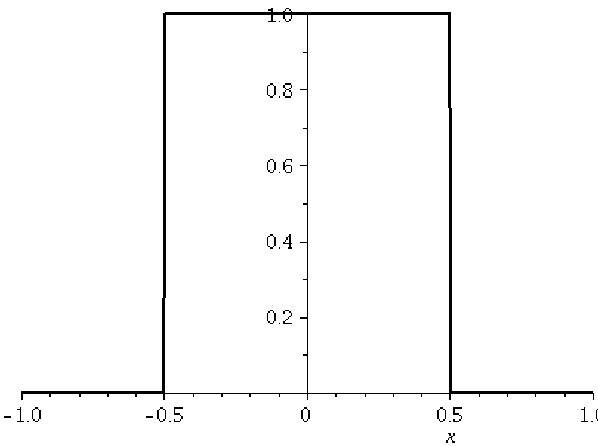
- The lower inequality means that if $\sum_k c[k]\varphi_k = 0$ then the coefficients *c* must be 0, so functions are linearly independent and uniquely defined by their coefficients.
- The upper bound means the function cannot be infinite and must be valid subspace of L_2.
- When A = B = 1, the basis function is orthonormal.

Sampling Functions

- The simplest function that satisfies the constraints is the box function:

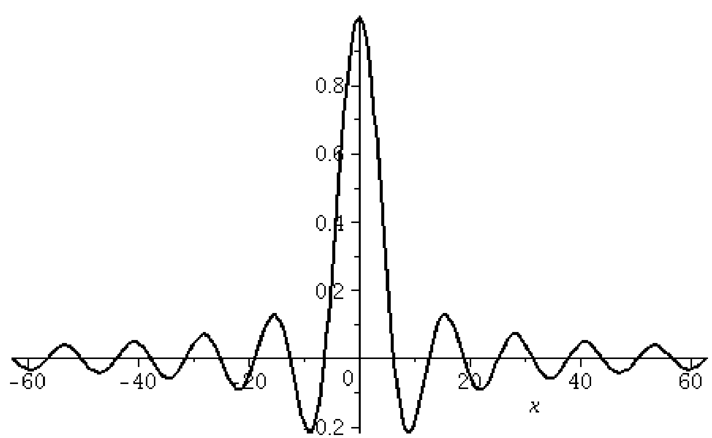
$$\varphi(x) = \mathrm{rect}(x) = \beta^0(x)$$

$$= \begin{cases} 1, & |x| < 1/2 \\ 0, & 1/2 < |x| \end{cases}$$



- The sinc function also satisfies the constraints by being orthogonal and partitioning unity:

$$\varphi(x) = \frac{\sin(x)}{x}$$



Sampling Functions

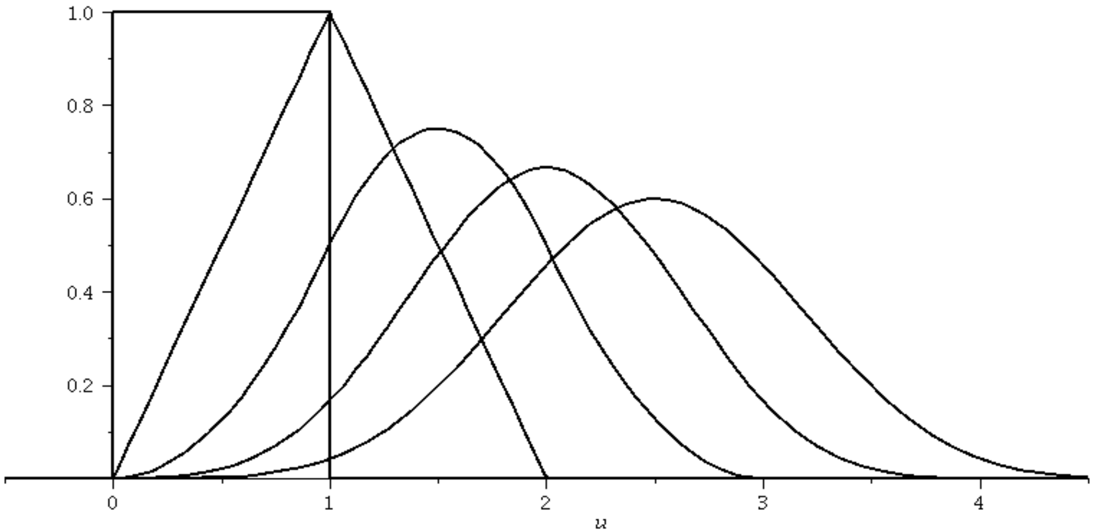
- Convolving the box with itself gives us the B-spline family of functions:

$$\beta^n(x) = \beta^0 * \beta^{n-1}(x), \qquad n \geq 1$$

B-splines are localized, not orthogonal yet satisfy the Riesz basis condition with:

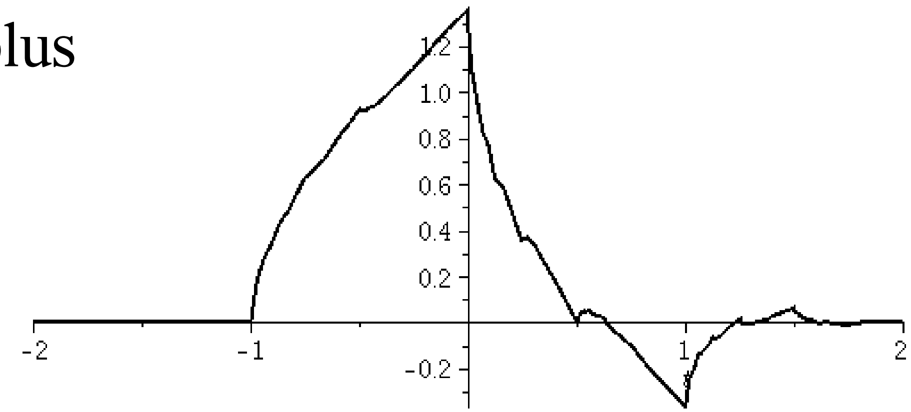
$$\Rightarrow \quad A \geq \left(\frac{2}{\pi}\right)^{n+1}, B = 1$$

and the partition of unity test.



Sampling Functions

- Any scaling function from Wavelet theory will satisfy the constraints (plus additional ones). For example the Daubechies D_4 wavelet:



- We can generate *equivalent* basis functions:

$$\varphi_{eq}(x) = \sum_{k \in \mathbb{Z}} p[k] \varphi(x-k)$$

if we can guarantee that $0 < C_1 \leq \left| P(e^{j\omega}) \right|^2 \leq C_2 < +\infty$

Sampling Functions

- We can also use *dual basis functions*, pairs of functions that are only orthogonal when used together:

$$\left\langle \varphi(x), \mathring{\varphi}(x-k) \right\rangle = \delta_k$$

- This observation leads us into the main idea...

Minimum Error Sampling

- How can we use these basis functions to faithfully approximate our input function $f(x) \in L_2$?
- We treat the problem as a *projection* of *f* onto our subspace V:

$$P_{V(\varphi)} f = \sum_{k \in \mathbb{Z}} \langle f(x), \mathring{\varphi}(x-k) \rangle \varphi(x-k)$$

where $\mathring{\varphi}(x)$ is the dual function of $\varphi(x)$:

$$\langle \mathring{\varphi}(x-l), \varphi(x-k) \rangle = \delta_{k-l}$$

- For $\mathring{\varphi}(x)$ to be in $V(\varphi)$ it must be a linear combination of $\varphi(x)$

$$\mathring{\varphi}(x) = \sum_{k \in \mathbb{Z}} p[k] \varphi(x-k)$$

Minimum Error Sampling

- Evaluating the inner product gives us:

$$\langle \mathring{\varphi}(x), \varphi(x-k)\rangle = \sum_{l\in\mathbb{Z}} p[l]\langle \varphi(x-l), \varphi(x-k)\rangle$$

$$= p * a_{\varphi}(k)$$

where a is the autocorrelation sequence.

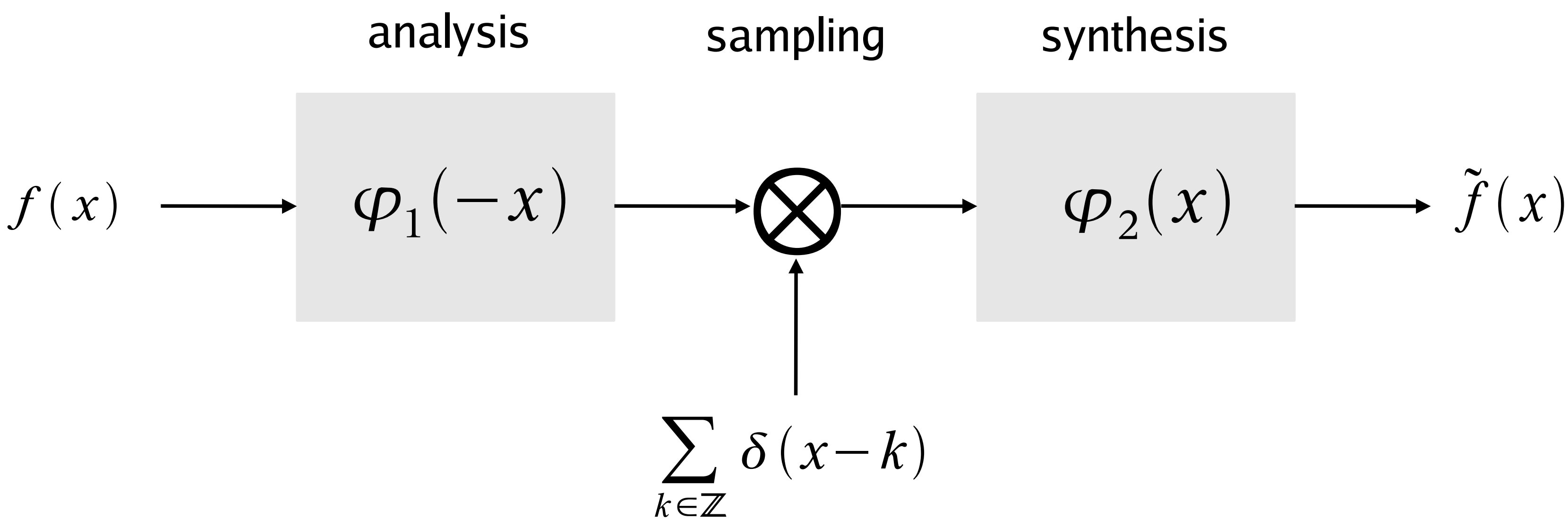
- By imposing the biorthogonality constraint from earlier:

$$p * a_{\varphi}(k) = \delta_k$$

we can solve for p in the Fourier space, and the Riesz condition guarantees that this is always possible.

Uh, what did we just do?

- Starting with a reconstruction filter, we showed that there is a *different* sampling filter that is optimal.



- These filters are no longer ideal, no longer orthogonal, yet sampled function can be exactly reconstructed without error.

Consistent Sampling in the Real World

- With most real-world devices we don't get to choose the sampling function. But we can still correctly reconstruct the signal.

- We cannot see the original signal, but we can insist on *consistency* - the ability to re-inject the final signal back into the system and get the same set of coefficients:

$$\forall f \in H, \quad c_1[k] = \langle f, \varphi_1(x-k) \rangle = \langle \tilde{f}, \varphi_1(x-k) \rangle$$

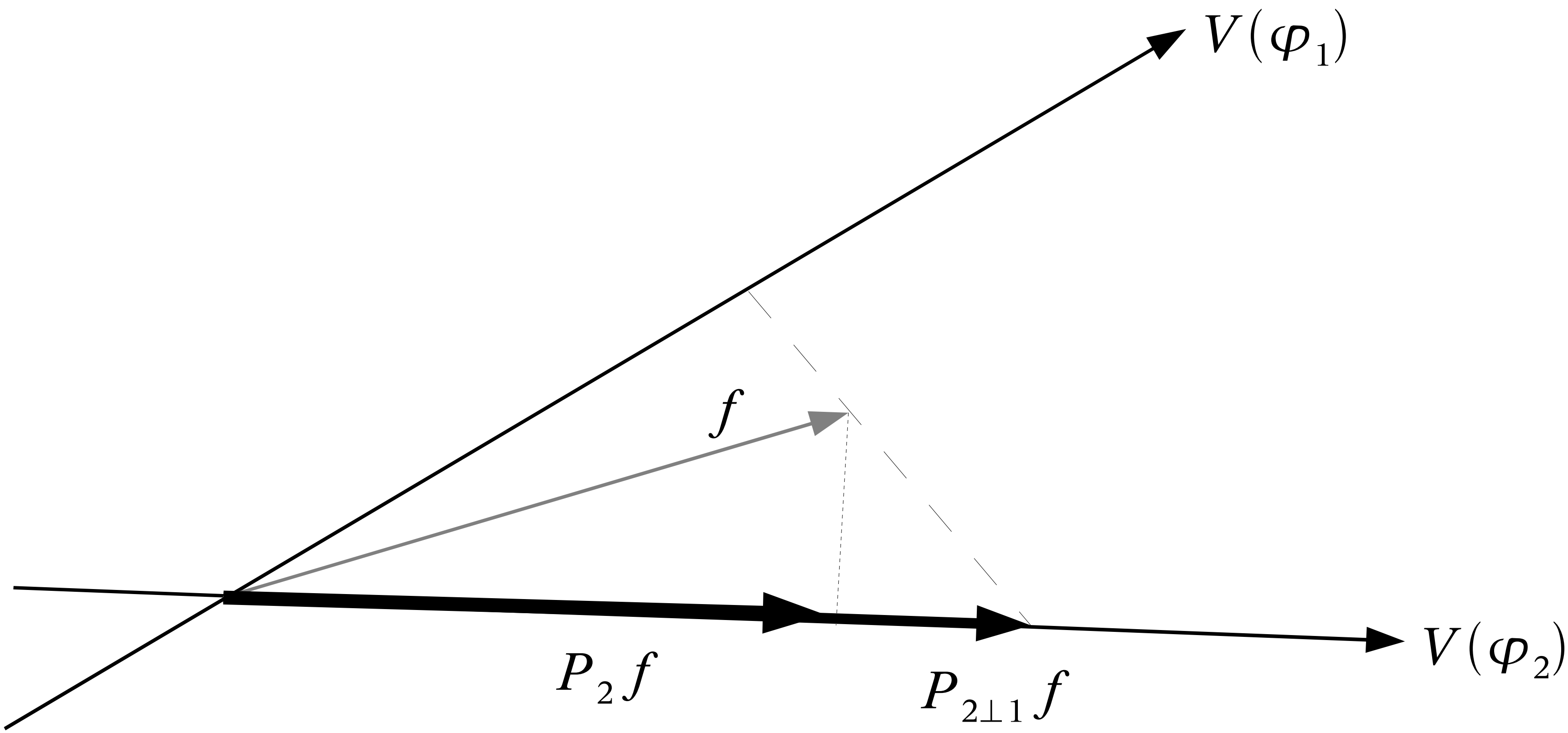
Allows us to add a filter q to the reconstruction function:

$$\tilde{f} = P_{2 \perp 1} f = \sum_{k \in Z} (q * c_1)[k] \varphi_2(x-k)$$

and we can solve for q in Fourier space.

Geometric Interpretation

- There is a geometric interpretation of this projection operator



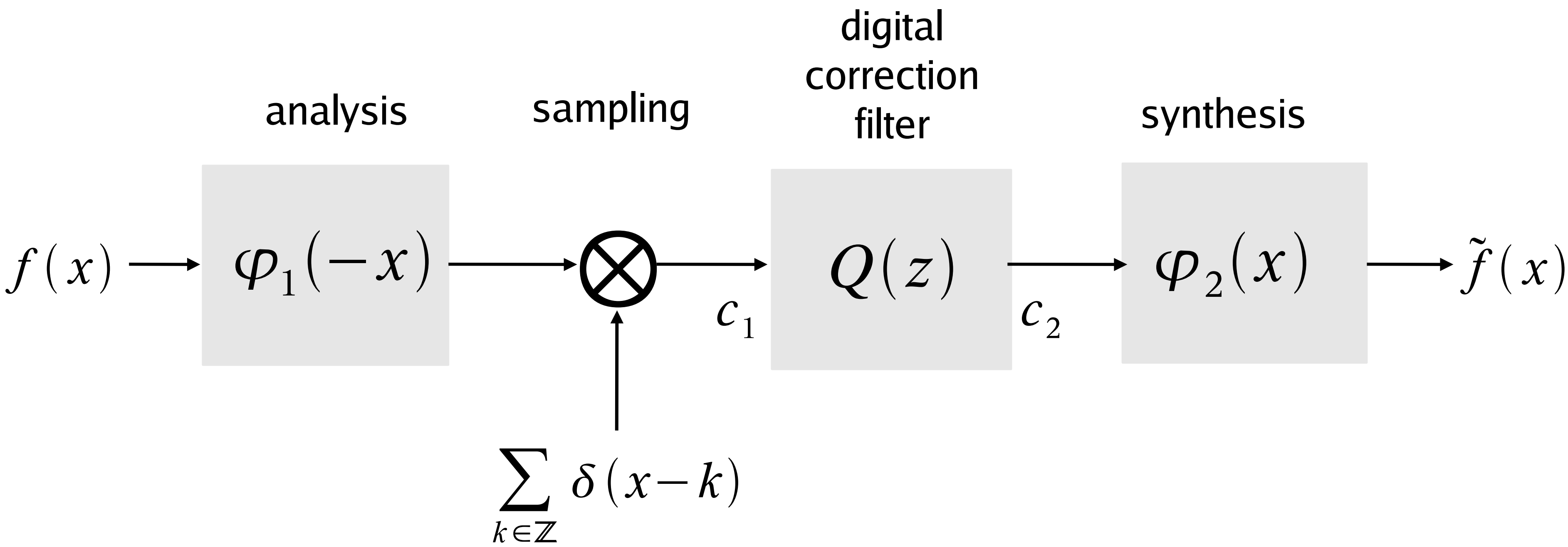
Generalization of Shannon's Theorem

- Every signal $f \in V(\varphi_2)$ can be reconstructed without error from it's samples.

- φ_1 and φ_2 can be selected freely.

- The functions need not be orthogonal.

Generalized Sampling



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